MOOC ‘From Big Bang to Dark Energy’ - Solution to Optional Challenge - Week 1

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1 Determine the points of closest and farthest approach for the Yukawa potential

Referring to Wikipedia: Specific orbital energy, the specific orbital energy equation combined with conservation of specific angular momentum for one of the points of closest/farthest approach (r_{close}, r_{far}) yields the total energy of the orbiting body of:

$$E_{total} = -\frac{k}{2A} \tag{1}$$

with $k = GMm$ and $A = \frac{r_{close}+r_{far}}{2}$ as the major semis-axis of the ellipse. Hence:

$$E_{total} = -\frac{k}{r_{close} + r_{far}} \tag{2}$$

1.1 Determine the velocity for the Newtonian potential using preservation of energy

Preservation of energy in the Newtonian potential states, using (2):

$$E_{kin} + E_{pot} = E_{total} \Leftrightarrow \frac{m}{2} \cdot v^2 - \frac{k}{r} = -\frac{k}{r_{close} + r_{far}}$$

Hence:

$$v^2 = \frac{2k}{m}(\frac{1}{r} - \frac{1}{r_{close} + r_{far}})$$

So:

$$v_{close}^2 = \frac{2k}{m}(\frac{1}{r_{close}} - \frac{1}{r_{close} + r_{far}})$$

Hence, after little transformation, for the closest point:

$$v_{close}^2 = \frac{2k}{m} \cdot \frac{r_{far}}{r_{close}(r_{close} + r_{far})} \tag{3}$$

Correspondingly for the farthest point:

$$v_{far}^2 = \frac{2k}{m} \cdot \frac{r_{close}}{r_{far}(r_{close} + r_{far})} \tag{4}$$

1.2 Determine the velocity for the Yukawa potential using preservation of angular momentum

Preservation of angular momemtum over Newtonian and Yukawa potential states that:

$$L'_{close} = L'_{far} = L' = L_{close} = L_{far} = L$$

equivalent to

$$m \cdot \vec{r'} \times \vec{v'} = m \cdot \vec{r} \times \vec{v} \Leftrightarrow m \cdot r \cdot v \cdot sin(\varphi') = m \cdot r \cdot sin(\varphi)$$

If we consider, for both cases, those points for which the radius and the velocity vector have the same angle to each other, that is, we assume that $sin(\varphi') = sin(\varphi)$, with $\varphi = \angle(\vec{r}, \vec{v})$, we obtain:

$$m \cdot r' \cdot v' = m \cdot r \Leftrightarrow r' \cdot v' = r \cdot v$$

Hence,

$$v' = \frac{r}{r'} \cdot v \tag{5}$$

for points on the orbit such that $\angle(\vec{r}, \vec{v}) = \angle(\vec{r'}, \vec{v'})$. Let's now consider the points on the major semi-axis, the points r_{close}, r_{far}. For these points, the angle beetween the radius vector and the velocity vector is $\frac{\pi}{2}$. We now assume that only for these points (provided the origin of the radius vector is shifted to the right or to the left from the center of the ellipse) it is the case that $\angle(\vec{r'}, \vec{v'}) = \frac{\pi}{2}$. Therefore, the points we obtain when applying the angular momementum preservation (3) to r_{close} and r_{far} are also the points on the major axis for the ellipse of the Yukawa potential, r'_{close} and r'_{far}, with their corressponding velocities v'_{close} and v'_{far}:

$$v'_{close} = \frac{r_{close}}{r'_{close}} \cdot v_{close} \tag{6}$$

$$v'_{far} = \frac{r_{far}}{r'_{far}} \cdot v_{far} \tag{7}$$

Using the equation for the squares of both velocities from equation (3) and (4), we obtain, for later use:

$$v'^2_{close} = \frac{2k}{m} \cdot \frac{r_{close} r_{far}}{(r_{close} + r_{far})} \cdot \frac{1}{r'^2_{close}} \tag{8}$$

$$v'^2_{far} = \frac{2k}{m} \cdot \frac{r_{close} r_{far}}{(r_{close} + r_{far})} \cdot \frac{1}{r'^2_{far}} \tag{9}$$

1.3 Determine points of closest and farthest approach for the Yukawa potential

Following the energy equation

$$E_{kin} + E_{pot} = E_{total}$$

we obtain for the Yukawa potential by equation its total engergy (LHS) with the total engergy of the Newtonian potential (RHS) assuming energy preservation:

$$\frac{m}{2}v^{2\prime} - \frac{k \cdot e^{-\frac{r'}{a}}}{r'} = -\frac{k}{r_{close} + r_{far}}$$

Substituting r' with x as placeholder for the closest and farhtest point of approach r'_{close} and r'_{far}, as well as the corresponding velocities from (8) and (9), we obtain one equation for both points:

$$\frac{k \cdot r_{close} r_{far}}{r_{close} + r_{far}} \frac{1}{x^2} - \frac{k \cdot e^{-\frac{x}{a}}}{x} + \frac{k}{r_{close} + r_{far}} = 0 \tag{10}$$

Cancelling k and multiplying both sides with x^2 yields

$$\frac{r_{close} r_{far}}{r_{close} + r_{far}} - xe^{-\frac{x}{a}} + \frac{k}{r_{close} + r_{far}} \cdot x^2 = 0$$

Taylor-expanding $e^{-\frac{x}{a}}$ upto the first-order term into $1 - \frac{x}{a}$ and rearranging terms, we obtain

$$(\frac{1}{r_{close} + r_{far}} + \frac{1}{a})x^2 - x + \frac{r_{close} r_{far}}{r_{close} + r_{far}} = 0 \tag{11}$$

We know, since r_{close} and r_{far} are located on the major semi-axis of the orbit ellipse, and both are π directed from each other, the sum of their length is twice the length of the major semi-axis A. Hence, since $r_{close} + r_{far} = 2A$, we arrive at the final quadratic equation:

$$(\frac{1}{2A} + \frac{1}{a})x^2 - x + \frac{r_{close} r_{far}}{2A} = 0 \tag{12}$$

Since we have introduced x as placeholder for the points of closest and farthest approach, using the solving formular for this quadratic equation has to yield r'_{close} and r'_{far}:

$$r'_{close} = x_1 = \frac{aA}{a + 2A} - \frac{\sqrt{a^2 A^2 - a^2 r_{close} r_{far} - 2aA r_{close} r_{far}}}{a + 2A} \tag{13}$$

$$r'_{far} = x_2 = \frac{aA}{a + 2A} + \frac{\sqrt{a^2 A^2 - a^2 r_{close} r_{far} - 2aA r_{close} r_{far}}}{a + 2A} \tag{14}$$

Plucking in the values given for Q1, $r_{close} = 1.4 \cdot 10^{11} m$, $r_{far} = 1.6 \cdot 10^{11} m$, $a = 1.5 \cdot 10^{15} m$ and $A = 1.5 \cdot 10^{11} m$, we get as solution for Q1 and Q2:

$$r'_{close} = 1.4019852692 \cdot 10^{11} m \tag{15}$$

$$r'_{far} = 1.5974148508 \cdot 10^{11} m \tag{16}$$

For later use, we calculate the major semi-axis A' for the Yukawa potential:

$$A' = \frac{r'_{close} + r'_{far}}{2} = \frac{a2A}{2(a + 2A)} \tag{17}$$

2 Determine the eccentricities for the Newtonian and the Yukawa potential

Using the general equation of an elliptical orbit given in the question

$$r = \frac{b}{1 + e \cdot cos(\Phi)} \tag{18}$$

and assuming Φ to be 0 and π at the points of closest and farthest approach, respectively, we obtain

$$r_{close} = \frac{b}{1 + e} \tag{19}$$

$$r_{far} = \frac{b}{1 - e} \tag{20}$$

For the Newtonian potential, we equal both b's:

$$r_{far} - r_{far} \cdot e = r_{close} + r_{close} \cdot e \Leftrightarrow r_{far} - r_{close} = e \cdot (r_{far} + r_{close})$$

Hence:

$$e = \frac{r_{far} - r_{close}}{r_{far} + r_{close}} \tag{21}$$

and correspondingly for the Yukawa potential:

$$e' = \frac{r'_{far} - r'_{close}}{r'_{far} + r'_{close}} \tag{22}$$

Plucking in the values, we obtain as solutions for Q3 and Q4

$$e = 0.0666667 \tag{23}$$

$$e' = 0.0651464 \tag{24}$$

3 Determine the difference beetween the eccentricities of both potentials in terms of a,b and e

Using $2A = r_{close} + r_{far}$ and equations (19) and (20), we can derive for the Newtonian potential like follows:

$$2A = \frac{b}{1 + e} + \frac{b}{1 - e} \Leftrightarrow \frac{b((1 - e) + (1 + e))}{(1 + e)(1 - e)} \Leftrightarrow \frac{2b}{1 - e^2}$$

Hence:

$$A = \frac{b}{1 - e^2} \Leftrightarrow A - Ae^2 = b$$

Hence:

$$e = \sqrt{1 - \frac{b}{A}} \tag{25}$$

and correspondingly for the Yukawa potential:

$$e' = \sqrt{1 - \frac{b}{A'}} \tag{26}$$

Considering (17) ($A' = \frac{a2A}{2(a+2A)}$), we obtain:

$$e' = \sqrt{1 - \frac{2b}{a2A}(a + 2A)} = \sqrt{1 - \frac{b}{A} - \frac{2b}{a}}$$

Substituting $x = \frac{b}{a}$, we get a function of x:

$$e' = f_{(x)} = (1 - \frac{b}{A} - 2x)^{\frac{1}{2}}$$

Since in the question it is defined that $a >> b$, it follows that $x \to 0$. Hence, we can perform a Taylor expansion. assuming $x_0 = 0$. Using the general form of the Taylor expansion around x_0

$$f_{(x)} = f_{(x_0)} + \frac{1}{1!}f'_{(x_0)}(x - x_0) + \frac{1}{2!}f''_{(x_0)}(x - x_0)^2 ...$$

and expanding upto the 1st derivate

$$f'_{(x)} = \frac{d(1 - \frac{b}{A} - 2x)^{\frac{1}{2}}}{dx} = -\frac{2}{2}(1 - \frac{b}{A} - 2x)^{-\frac{1}{2}} \tag{27}$$

so that

$$f'_{(x_0=0)} = -(1 - \frac{b}{A})^{-\frac{1}{2}} = -\frac{1}{\sqrt{1 - \frac{b}{A}}} \tag{28}$$

Together with $f_{(x_0=0)} = \sqrt{1 - \frac{b}{A}}$, we obtain as approximation of the eccentricity for the Yukawa potential:

$$e' = f_{(x)} = \sqrt{1 - \frac{b}{A}} - \frac{x}{\sqrt{1 - \frac{b}{A}}} \tag{29}$$

We notice the term $\sqrt{1 - \frac{b}{A}}$ being indentical to e. Hence:

$$e' = f_{(x)} = e - \frac{x}{e}$$

Building the difference $\delta e = e' - e$, we finally get as answer for Q5:

$$\delta e = -\frac{x}{e} = -\frac{b}{ae} \tag{30}$$

In order to semi-verify this result, we let the variable a go to infinity, leading back to the Newtonian potential. The difference in the eccentricity has to vanish:

$$\lim_{a \to \infty} \delta e = 0 \tag{31}$$

Q.E.D.